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Unconditioned strong d-sequences

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A sequence a_1, a_2, \dots, a_s of elements in a commutative ring A is called a d-sequence if the equality

$$[(a_1, \dots, a_{i-1}): a_i a_j] = [(a_1, \dots, a_{i-1}): a_j]$$

holds for all $1 \leq i \leq j \leq s$ and is called a strong d-sequence if $a_1^{n_1}, \dots, a_s^{n_s}$ is a d-sequence for all $n_1, \dots, n_s > 0$. Moreover if a_1, a_2, \dots, a_s is a (strong) d-sequence in any order, we will say that a_1, a_2, \dots, a_s is an unconditioned (strong) d-sequence ([9]).

The behaviour of parameter ideals in Buchsbaum rings are studied by S. Goto and many others (cf., [5] and [6]). The aim of this lecture is to give a new approach to them in terms of unconditioned strong d-sequences. This motif is suggested by Prof. S. Goto, and the writer wishes to thank him and the members of his seminar for helpful suggestions.

1. Let a_1, a_2, \dots, a_s be a sequence of elements in a commutative ring A and put $I = (a_1, \dots, a_s)$. Our first result is stated as follows

Theorem 1. Let a_1, a_2, \dots, a_s be an unconditioned strong d-sequence. Then the following equalities hold:

$$(1) \quad [(a_1^{n_1+m_1}, \dots, a_s^{n_s+m_s}): \prod_{i=1}^s a_i^{m_i}] = \sum_{i=1}^s [(a_1^{n_1}, \dots, \widehat{a_i^{n_i}}, \dots, a_s^{n_s}): a_i] + (a_1^{n_1}, \dots, a_s^{n_s}) \quad \text{for all } n_i, m_i > 0 \ (1 \leq i \leq s);$$

$$(2) \quad (a_1^{n_1}, \dots, a_k^{n_k}) \cap I^n = \sum_{i=1}^k a_i^{n_i} \cdot I^{n-n_i} \quad \text{for all}$$

$1 \leq k \leq s$, $n_1, \dots, n_k > 0$ and $n \in \mathbb{Z}$;

$$(3) \quad \left[(a_1^{n_1}, \dots, a_{k-1}^{n_{k-1}}) : a_k \right] \cap I^n = \sum_{i=1}^{k-1} a_i^{n_i} \cdot I^{n-n_i} + \sum_{\substack{F \subset \{1, \dots, k-1\} \\ \sum_{f \in F} (n_f - 1) \geq n}} \prod_{f \in F} a_f^{n_f - 1} \cdot [(a_f | f \in F) : a_k]$$

for all $1 \leq k \leq s$, $n_1, \dots, n_{k-1} > 0$ and $n \in \mathbb{Z}$.

Let $G_I(A) = \bigoplus_{n \geq 0} I^n / I^{n+1}$ and $R_I(A) = \bigoplus_{n \geq 0} I^n$ denote the associated graded ring of I and the Rees algebra of I , respectively. $H_I^i(\cdot)$ (resp. $H_{\underline{M}}^i(\cdot)$ and $H_{\underline{N}}^i(\cdot)$) stands for the i^{th} local cohomology defined by the direct limit of Koszul cohomology relative to I (resp. $\underline{M} = G_I(A)_+$ and $\underline{N} = I \cdot R_I(A) + R_I(A)_+$). With this terminology we also have the following

Theorem 2. Let a_1, a_2, \dots, a_s be an unconditioned strong d -sequence. Then

(1) $I \cdot H_I^i(A / (a_1^{n_1}, \dots, a_j^{n_j})) = (0)$ for all $0 \leq i + j < s$ and $n_1, \dots, n_j > 0$.

$$(2) \quad \begin{aligned} [H_{\underline{M}}^i(G_I(A))]_n &= H_I^i(A) \quad (n = -i), \\ &= (0) \quad (n \neq -i) \end{aligned}$$

for every $0 \leq i < s$ and

$$[H_{\underline{M}}^s(G_I(A))]_n = (0) \quad (n > -s).$$

$$(3) \quad \begin{aligned} [H_{\underline{N}}^0(R_I(A))]_n &= H_I^0(A) \quad (n = 0), \\ &= (0) \quad (n \neq 0), \end{aligned}$$

$$\begin{aligned} [H_{\underline{N}}^i(R_I(A))]_n &= H_I^{i-1}(A) \quad (2 - i \leq n \leq -1), \\ &= (0) \quad (n \leq 1 - i \text{ or } n \geq 0) \end{aligned}$$

for every $1 \leq i \leq s$ and

$$[H_{\underline{N}}^{s+1}(R_I(A))]_n = (0) \quad (n \geq 0).$$

$$(4) \quad H_{\underline{N}}^i(R_I(A)) = (0) \quad \text{for all } i > s + 1.$$

All conclusions of our theorems are led by Lemma 4 which is given by S. Goto and another powerful application of this lemma can be found in [18]. So we believe that this lemma plays very important roles in the whole of our research.

In order to discuss an application of our results, let us recall some definition. For a while let A be a Noetherian local ring of $\dim A = d > 0$ and m the maximal ideal of A . Then A is called a Buchsbaum ring if the difference

$$l_A(A/q) - e_A(q)$$

is an invariant $I(A)$ of A not depending on the particular choice of a parameter ideal q of A , where $l_A(A/q)$ and $e_A(q)$ denote the length of the A -module A/q and the multiplicity of A relative to q , respectively. This is equivalent to saying that every system a_1, a_2, \dots, a_d of parameters for A is a weak-sequence, i.e., the equality $[(a_1, \dots, a_{i-1}): a_i] = [(a_1, \dots, a_{i-1}): m]$ holds for all $1 \leq i \leq d$ ([14]). C. Huneke showed in [9, (1.7)] that A is Buchsbaum if and only if every system of parameters for A forms a d -sequence. The theory of Buchsbaum rings (and modules) has rapidly developed and nowadays much is known about them (cf., e.g., [1], [3], [4], [7], [8], [10], [11], [12], [13], [15], [16], [20]). Let q be a parameter ideal of A and put

$$G_q(A) = \bigoplus_{n \geq 0} q^n / q^{n+1} \quad \text{and} \quad R_q(A) = \bigoplus_{n \geq 0} q^n$$

, the associated graded ring of q and the Rees algebra of q , respectively. As an application of our results we give an affirmative answer to the question posed by S. Goto in [2] as follows: is $R_q(A)$ a Buchsbaum ring if so is A ? Our answer

is stated as follows

Theorem 3. Suppose that $\text{depth } A > 0$. Then the following conditions are equivalent.

- (1) A is a Buchsbaum ring;
- (2) $G_q(A)_{\underline{M}}$ is a Buchsbaum ring for every parameter ideal q of A ;
- (3) $R_q(A)_{\underline{N}}$ is a Buchsbaum ring for every parameter ideal q of A ,

here \underline{M} and \underline{N} denote the unique graded maximal ideal of $G_q(A)$ and $R_q(A)$ respectively.

2. Sketch of Proof of Theorem 1.

Let a_1, a_2, \dots, a_s be a sequence of elements in a commutative ring A and put $I = (a_1, \dots, a_s)$. Then

Lemma 4 (S. Goto). If a_1, a_2, \dots, a_s is an unconditioned strong d -sequence modulo bA , then the equality

$$[(a_1^{n_1}, \dots, a_s^{n_s}) : b] = \sum_{F \subset \{1, \dots, s\}} \prod_{f \in F} a_f^{n_f - 1} \cdot [(a_f |_{f \in F}) : b]$$

holds for all $n_1, \dots, n_s > 0$.

Lemma 5 ([6, Theorem (2.4)]). If a_1, a_2, \dots, a_s is a d -sequence, then

$$[(a_1, \dots, a_{i-1}) : a_i] \cap I^n = (a_1, \dots, a_{i-1}) \cdot I^{n-1}$$

for all $1 \leq i \leq s$ and $n > 0$.

Lemma 6. If a_1, a_2, \dots, a_s is an unconditioned strong d -sequence, then the following conditions are equivalent.

$$(1) \quad (a_f^{n_f} | f \in F) \cap I^n = \sum_{f \in F} a_f^{n_f} \cdot I^{n-n_f}$$

for all $F \subset \{1, \dots, s\}$, $n_f > 0$ ($f \in F$) and $n \in \mathbb{Z}$.

$$(2) \quad (a_f^{n_f} \mid f \in F) \bigcap I^n = \sum_{f \in F} a_f^{n_f} \cdot I^{n-n_f}$$

for all $F \subset \{1, \dots, s\}$, $n_f = 1, 2$ ($f \in F$) and $n \in \mathbb{Z}$.

$$(3) \quad (a_1^{n_1}, \dots, a_s^{n_s}) \bigcap I^n = \sum_{i=1}^s a_i^{n_i} \cdot I^{n-n_i}$$

for all $n_i = 1, 2$ ($1 \leq i \leq s$) and $n \in \mathbb{Z}$.

Lemma 7. If a_1, a_2, \dots, a_s is an unconditioned strong d-sequence such that

$$(a_1^2, \dots, a_{s-1}^2, a_s) \bigcap I^n = (a_1^2, \dots, a_{s-1}^2) \cdot I^{n-2} + a_s \cdot I^{n-1}$$

for all $n \in \mathbb{Z}$, then the equality

$$\begin{aligned} & \left([(a_1^2, \dots, a_{s-1}^2) : a_s] + a_s A \right) \bigcap I^n = (a_1^2, \dots, a_{s-1}^2) \cdot I^{n-2} \\ & + a_s \cdot I^{n-1} + \sum_{\substack{F \subset \{1, \dots, s-1\} \\ \#F \geq n}} \prod_{f \in F} a_f \cdot [(a_f \mid f \in F) : a_s] \end{aligned}$$

holds for all $n \in \mathbb{Z}$.

Proof of Theorem 1. (1) Use induction on s and apply

Lemma 4. (2) By Lemma 6, we must show that

$$(a_1^{n_1}, \dots, a_s^{n_s}) \bigcap I^n = \sum_{i=1}^s a_i^{n_i} \cdot I^{n-n_i}$$

for all $n_i = 1, 2$ ($1 \leq i \leq s$) and $n \in \mathbb{Z}$. Case 1: $n_i = 1$ for some i . Use induction on s and apply Lemma 5. Case 2: $n_i = 2$ for all i . Apply Lemma 7 and Case 1. (3) Use induction on n and apply Lemma 4 and the above assertion (2).

3. Sketch of Proof of Theorem 2.

Let J be an ideal of A . A sequence a_1, a_2, \dots, a_s is called a weak-sequence with respect to J if the equality

$$[(a_1, \dots, a_{i-1}) : a_i] = [(a_1, \dots, a_{i-1}) : J]$$

holds for all $1 \leq i \leq s$. We will use the words "strong" and "unconditioned" with the same meaning as d-sequences.

Lemma 8. Let J be an ideal of A which contains I . Then a_1, a_2, \dots, a_s is an unconditioned strong weak-sequence with respect to J if and only if it is an unconditioned strong d -sequence and an unconditioned weak-sequence w.r.t. J .

Proof of (1) of Theorem 2. By Lemma 8, we see that

$$I. H_I^0(A/(a_1^{n_1}, \dots, a_j^{n_j})) = (0)$$

for every $0 \leq j < s$, $n_1, \dots, n_j > 0$. The assertion comes at once by descending induction on j and the similar discussion as in [20, Lemma 2].

We put $h_i = a_i \bmod I^2$ ($1 \leq i \leq s$). Then

Proposition 9. h_1, h_2, \dots, h_s is an unconditioned strong d -sequence if so is a_1, a_2, \dots, a_s .

Lemma 10. If a_1, a_2, \dots, a_s is an unconditioned strong d -sequence, then

$$(1) \quad \begin{aligned} [H_M^0(G_I(A))]_n &= H_I^0(A) \quad (n = 0), \\ &= (0) \quad (n \neq 0); \end{aligned}$$

$$(2) \quad [H_M^i(G_I(A))]_n = (0) \quad (n \neq -i)$$

for every $1 \leq i < s$;

$$(3) \quad [H_M^s(G_I(A))]_n = (0) \quad \text{for all } n > -s.$$

We put $a = a_1$. As $aA \cap I^n = a \cdot I^{n-1}$ ($n \in \mathbb{Z}$) by Theorem 1, we get the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & [0 : a] & \longrightarrow & R_I(A) & \xrightarrow{a} & R_I(A) \longrightarrow R_I(A)/a \cdot R_I(A) \longrightarrow 0 \\ & & & & \searrow & \nearrow & \uparrow \\ & & & & & a \cdot R_I(A) & R_I(A)/aA \uparrow (A/aA)_+ \\ & & & & \nearrow & \searrow & \uparrow \\ & & & & 0 & & 0 \end{array}$$

Lemma 11. If a_1, a_2, \dots, a_s is an unconditioned strong d -sequence, then

- (1)
$$\begin{aligned} [H_{\underline{N}}^0(R_I(A))]_n &= H_I^0(A) \quad (n = 0), \\ &= (0) \quad (n \neq 0); \end{aligned}$$
- (2) the canonical map $H_{\underline{N}}^0(R_I(A)) \longrightarrow H_{\underline{N}}^0(R_I(A)/a \cdot R_I(A))$ is an isomorphism;
- (3)
$$\begin{aligned} [H_{\underline{N}}^i(R_I(A))]_n &= (0) \quad (n \leq -i \text{ or } n \geq 0) \text{ for every } 1 \leq i \leq s \\ \text{and } [H_{\underline{N}}^{s+1}(R_I(A))]_n &= (0) \quad (n \geq 0). \end{aligned}$$

Proof of Theorem 2. Consider the following two exact sequences:

$$\begin{aligned} 0 &\longrightarrow R_I(A)_+ \longrightarrow R_I(A) \longrightarrow A \longrightarrow 0, \\ 0 &\longrightarrow R_I(A)_+(1) \longrightarrow R_I(A) \longrightarrow G_I(A) \longrightarrow 0. \end{aligned}$$

Use these sequences, and apply Lemma 10 and 11.

4. Sketch of Proof of Theorem 3.

Let A be a Noetherian local ring of $\dim A = d > 0$ and \mathfrak{m} the maximal ideal of A . We say that A has finite local cohomology if the local cohomology $H_{\mathfrak{m}}^i(A)$ of A are finitely generated (i.e., the length $l_A(H_{\mathfrak{m}}^i(A))$ are finite) for all $i \neq d$. Let a_1, a_2, \dots, a_d be a system of parameters for A and put $q = (a_1, \dots, a_d)$. We define that

$$I(a_1, \dots, a_d; A) = l_A(A/q) - e_A(q)$$

, where $l_A(A/q)$ and $e_A(q)$ denote the length of an A -module A/q and the multiplicity of A relative to q , respectively.

Lemma 12. The following conditions are equivalent.

- (1) a_1, a_2, \dots, a_d is an unconditioned strong d -sequence.
- (2) $I(a_1, \dots, a_d; A) = I(a_1^2, \dots, a_d^2; A)$.

(3) A has finite local cohomology and

$$I(a_1, \dots, a_d; A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \cdot l_A(H_m^i(A)) .$$

Let G denote the associated graded ring $G_q(A) = \bigoplus_{n \geq 0} q^n / q^{n+1}$ of q and h_i the initial form of a_i ($1 \leq i \leq d$).

Lemma 13. h_1, h_2, \dots, h_d is an unconditioned strong d -sequence if and only if so is a_1, a_2, \dots, a_d .

In the 4th Symposium on Commutative Algebra at Karuizawa in Japan (Nov. 3-6, 1982), N. V. Trung introduces a standard system of parameters for an A -module E which is a system a_1, a_2, \dots, a_d of parameters for E satisfying the same condition as (3) (and (2)) of Lemma 12, i.e., $H_m^i(E)$ is finitely generated for all $i \neq \dim_A E$ and $l_A(E/(a_1, \dots, a_d).E) = e_E(a_1, \dots, a_d) = \sum_{i=0}^{d-1} \binom{d-1}{i} \cdot l_A(H_m^i(E))$, where $d = \dim_A E$. Hence a system of parameters for A is an unconditioned strong d -sequence if and only if it is standard ([19]).

Let us recall that A is called a quasi-Buchsbaum ring if $m \cdot H_m^i(A) = (0)$ for all $i \neq d$. This is equivalent saying that at least one (and hence every) system of parameters for A ^{in m^2} is a weak-sequence (By a weak-sequence we mean a weak-sequence with respect to the maximal ideal of the local ring A), see ([17]).

Proposition 14 ([5], also [19]). The following two conditions are equivalent.

- (1) $G_{\underline{M}}$ is a Buchsbaum ring, where $\underline{M} = m.G + G_+$.
 - (2) a_1, a_2, \dots, a_d is an unconditioned strong weak-sequence.
- In this case, A is a quasi-Buchsbaum ring and $I(G_{\underline{M}}) = I(A)$.

Theorem 15. Let A be a quasi-Buchsbaum ring. Then the following two conditions are equivalent.

- (1) $G_{\underline{M}}$ is a quasi-Buchsbaum ring with $I(G_{\underline{M}}) = I(A)$.
- (2) The equality

$$(a_1^2, \dots, a_d^2) \cap q^n = \sum_{i=1}^d a_i^2 \cdot q^{n-2}$$

holds for all $n \in \mathbb{Z}$.

Lemma 16. Suppose that $\text{depth } A > 0$ and $\text{Proj } R_q(A)$ is Cohen-Macaulay. If a_1, a_2, \dots, a_{d-1} is an unconditioned strong d -sequence modulo $a_d^n A$ for almost all $n > 0$, then the equality

$$[(a_1, \dots, a_{d-1}) : a_d^2] = [(a_1, \dots, a_{d-1}) : a_d]$$

holds.

Proposition 17 ([6]). Suppose that $\text{depth } A > 0$. Then the following two conditions are equivalent.

- (1) a_1, a_2, \dots, a_d is an unconditioned strong d -sequence.
- (2) $\text{Proj } R_{(a_1^{n_1}, \dots, a_d^{n_d})}(A)$ is Cohen-Macaulay for all $n_1, \dots, n_d > 0$.

Proof of Theorem 3. The equivalence of (1) and (2) comes at once from Proposition 14. (3) \implies (1) follows from Proposition 17 and the converse is proved by Surjective Criterion in [13] and [15].

Finally we also have the following

Proposition 18 ([5]). If a_1, a_2, \dots, a_d is an unconditioned strong d -sequence, then

$$e_A(q) \geq \sum_{i=1}^{d-1} \binom{d-1}{i-1} \cdot l_A(H_m^i(A)) .$$

References

- [1] S. Goto, On Buchsbaum rings, J. Alg., 67(1980), 272-279.
- [2] _____, Blowing-up characterization for local rings, R.I. M.S. Kôkyûroku, 400(1980), 42-50.
- [3] _____, Buchsbaum rings with multiplicity 2, J. Alg., 74(1982), 494-508.
- [4] _____, Buchsbaum rings of maximal embedding dimension, J. Alg., 76(1982), 383-399.
- [5] _____, On the associated graded rings of parameter ideals in Buchsbaum rings, to appear in J. Alg.
- [6] _____, Blowing-up of Buchsbaum rings, to appear in The Proc. Durham Symposium on Commutative Algebra.
- [7] _____, Noetherian local rings with Buchsbaum associated graded rings, to appear in J. Alg.
- [8] S. Goto and Y. Shimoda, On Rees algebras over Buchsbaum rings, J. Math. Kyoto Univ., 20(1980), 691-708.
- [9] C. Huneke, The theory of d-sequences and powers of ideals, Ad. Math., 46(1982), 249-279.
- [10] S. Ikeda, Cohen-Macaulayness of Rees algebras of local rings, preprint.
- [11] B. Renschuch, J. Stückrad and W. Vogel, Weitere Bemerkungen zu einem Problem der Schnitttheorie und über ein Maß von A. Seidenberg für die Imperfektheit, J. Alg., 37(1975), 447-471.
- [12] P. Schenzel, Applications of dualizing complexes to Buchsbaum rings, Ad. Math., 44(1982), 61-77.
- [13] J. Stückrad, Über die kohomologische Charakterisierung von Buchsbaum-Moduln, Math. Nachr., 95(1980), 265-272.
- [14] J. Stückrad and W. Vogel, Eine Verallgemeinerung der Cohen-Macaulay Ringe und Anwendungen auf ein Problem ^{der} Multiplizitätstheorie, J. Math. Kyoto Univ., 13(1973), 513-528.
- [15] _____, Toward a theory of Buchsbaum singularities, Amer. J. Math., 100(1978), 727-746.
- [16] N. Suzuki, On the Koszul complex generated by a system of parameters for a Buchsbaum module, I and II, the Bulletin of Department of General Education of Sizuoka College of

Pharmacy, 8(1979), 27-38 and 10(1981), 67-70.

- [17] N. Suzuki, On a basic theorem for quasi-Buchsbaum modules, the Bulletin of Department of General Education of Shizuoka College of Pharmacy, 11(1982), 33-40.
- [18] _____, Canonical duality for Buchsbaum modules — An application of Goto's lemma on Buchsbaum modules, in preparation.
- [19] N. V. Trung, Standard systems of parameters of generalized Cohen-Macaulay modules, Report of the 4th Symposium on Commutative Algebra, at Karuizawa in Japan, Nov. 3-6, 1982, to appear.
- [20] W. Vogel, A nonzero-divisor characterization of Buchsbaum modules, Michigan Math. J., 28(1981), 147-152.